Helmholtz Decomposition and Rotation Potentials in n-dimensional Cartesian Coordinates

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Abstract: This paper introduces a novel method to extend the Helmholtz Decomposition to n-dimensional sufficiently smooth and fast decaying vector fields. The rotation is described by a superposition of n(n+1)/2 rotations within the coordinate planes. The source potential and the rotation potential are obtained by convolving the source and rotation densities with the fundamental solutions of the Laplace equation. The rotation-free gradient of the source potential and the divergence-free rotation of the rotation potential sum to the original vector field. The approach relies on partial derivatives and Newton integrals and allows for a simple application of this standard method to high-dimensional vector fields, without using concepts from differential geometry and tensor calculus.

Keywords: Helmholtz Decomposition, Fundamental Theorem of Calculus, Curl Operator

1. Introduction

The Helmholtz Decomposition splits a sufficiently smooth and fast decaying vector field into an irrotational (curl-free) and a solenoidal (divergence-free) vector field. In $\mathbb{R}^3$, this 'Fundamental Theorem of Vector Calculus' allows to calculate a scalar and a vector potential that serve as antiderivatives of the gradient and the curl operator. This tool is indispensable for many problems in mathematical physics (Dassios and Lindell, 2002; Kustepeli, 2016; Sprössig, 2009; Suda, 2020), but has also found applications in animation, computer vision, robotics (Bhatia et al., 2013), or for describing a 'quasi-potential' landscapes and Lyapunov functions for high-dimensional non-gradient systems (Suda, 2019; Zhou et al., 2012). The literature review in Section 2 summarizes the classical Helmholtz Decomposition in $\mathbb{R}^3$ and the Helmholtz–Hodge Decomposition that generalizes the operator curl to higher dimensions using concepts of differential geometry and tensor calculus.

Section 3 introduces a much simpler generalization to higher dimensions in Cartesian coordinates and states the Helmholtz Decomposition Theorem using novel differential operators. Section 4 defines these operators to derive a source density and a rotational density describing the $\binom{n}{2}$ basic rotations within the 2-dimensional coordinate planes. By Newton Integration, a scalar source potential and $\binom{n}{2}$ rotation potentials can be obtained, and the original vector can be decomposed into a rotation-free vector and a divergence-free vector by superposing the gradient of the source potential with the rotation of the rotation potentials. Section 5 states some propositions and proves the theorem. Section 6 concludes.

2. Literature review

2.1. Classical Helmholtz Decomposition in $\mathbb{R}^3$

In its classical formulation,[1] the Helmholtz Decomposition decomposes a vector field $f \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ that decays faster than $1/|x|$ for $|x| \to \infty$ into an irrotational (curl-free) vector field $g(x) = -\operatorname{grad}\Phi(x)$ with a scalar potential $\Phi \in C^3(\mathbb{R}^3, \mathbb{R})$ and a solenoidal (divergence-free) vector field $r(x) = \operatorname{curl} A(x)$ with a vector potential $A \in C^3(\mathbb{R}^3, \mathbb{R}^3)$ such that $f(x) = g(x) + r(x)$.

The potentials can be derived by calculating the source density $\gamma(x)$ and the rotation density $\rho(x)$:

$$\gamma(x) = \operatorname{div} f(x) = \operatorname{div} g(x), \qquad \rho(x) = \operatorname{curl} f(x) = \operatorname{curl} r(x). \tag{1}$$

The convolution with the fundamental solutions of the Laplace equation provides the potentials:[2]

$$\Phi(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\gamma(\xi)}{|x-\xi|} d\xi^3, \qquad A(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\rho(\xi)}{|x-\xi|} d\xi^3. \tag{2}$$

The Helmholtz decomposition of f is given as:

$$f(x) = g(x) + r(x) \quad \text{with} \quad g(x) = -\operatorname{grad}\Phi(x) \quad \text{and} \quad r(x) = \operatorname{curl} A(x). \tag{3}$$

2.2. Previous extensions to higher dimensions

For generalizing the Helmholtz Decomposition to higher-dimensional manifolds, divergence and gradient can straightforwardly be extended to any dimension n, but not the operator curl and the cross product. This lead to the Hodge Decomposition within the framework of differential forms, defining the operator curl as the Hodge dual of the anti-symmetrized gradient (Hauser, 1970; McDavid and McMullen, 2006; Tran-Cong, 1993; Vargas, 2014).

In two Cartesian dimensions, curl acting on a scalar field R is a two-dimensional vector field given by

$$\operatorname{curl} R(x) = \left[-\frac{\partial R}{\partial x_2}, \frac{\partial R}{\partial x_1}\right] \quad = \left[\delta_{2k}\frac{\partial R}{\partial x_1} - \delta_{1k}\frac{\partial R}{\partial x_2}; 1 \le k \le 2\right]. \tag{4}$$

Square brackets indicate vectors in $\mathbb{R}^n$. The second notation will help to compare this rotation in the x_1-x_2-plane with the rotation in the x_i-x_j-plane in Eq. (29). The rotation operator acting on a two-dimensional vector field f yields a scalar field given by:

$$\overline{\operatorname{curl}} f(x) = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \quad = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \text{ with } i = 1, j = 2. \tag{5}$$

Here, we use the overline to indicate that $\overline{\operatorname{curl}}$ is operating on vector fields, and curl without overline operates on scalar fields. For $n = 3$, the rotation of a vector field f is usually written as a pseudovector:

$$\overline{\operatorname{curl}} f(x) = \left[\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right]. \tag{6}$$

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1 For a historical overview of the contributions by Stokes (1849) and Helmholtz (1858), see Kustepeli (2016).

2 The restriction to fast decaying fields has been relaxed by Blumenthal (1905), Gurtin (1962), Gregory (1996), Petrascheck (2015) and most strongly by Tran-Cong (1993) showing that $f(x)$ needs to be bounded at infinity only by $O(|x|^l)$ with l a constant using a more complicated convolution integral.

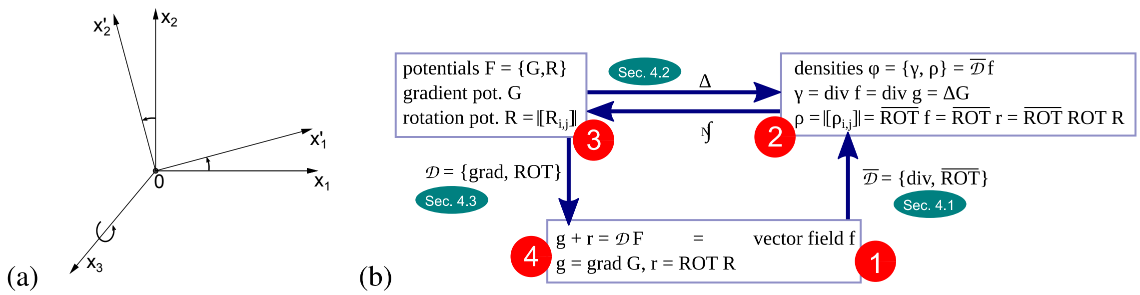

Figure 1: (a) In $\mathbb{R}^3$, a rotation around the x_3-axis corresponds to a rotation within the x_1-x_2-plane. (b) Relation between source density ϕ, force f and Newton potential F in $\mathbb{R}^n$. The operators are described in the sections indicated in the petrol ellipses.

The third component (and analogously first and second) is often understood as the rotation *around* the x_3-coordinate. In order to facilitate the extension to higher dimensions, it should better be discussed as rotation *within* the x_1-x_2-coordinate plane, see Figure 1(a). Then, curl should be understood as an antisymmetric second rank tensor (Gonano and Zich, 2014; McDavid and McMullen, 2006). There exist $\binom{n}{2}$ rotations within the coordinate planes. Only for $n = 3$ can each of these rotations be described as a rotation around a vector, as $\binom{n}{2} = n$ if and only if $n = 3$.

For a tensor field T with dimension $n > 3$ and rank k, curl T is a tensor field with dimension n and rank $n - k - 1$. For a scalar (rank $k = 0$), it consists of n^{n-1} entries. For a vector field in $\mathbb{R}^n$ (rank $k = 1$), $\overline{\text{curl}}$ consists of n^{n-2} components. Each component needs $n - 2$ indices and is given in Cartesian coordinates by:

$$\left(\overline{\text{curl}} f\right)_{e_1,\dots,e_{n-2}} = \sum_{1\le l,m\le n} \tfrac{1}{2}\epsilon_{e_1,\dots,e_{n-2},l,m} \cdot \left(\frac{\partial f_l}{\partial x_m} - \frac{\partial f_m}{\partial x_l}\right) = \sum_{1\le l,m\le n} -\epsilon_{e_1,\dots,e_{n-2},l,m} \cdot \frac{\partial f_l}{\partial x_m}, \tag{7}$$

with the completely antisymmetric Levi–Civita tensor $\epsilon_{\nu_1\dots\nu_p}$ that is $+1$ (resp. -1) if the integers $\nu_1 \dots \nu_n$ are distinct and an even (resp. odd) permutation of $1 \dots n$, and otherwise 0.

As an example, for $\mathbb{R}^5$, the term $\frac{\partial f_5}{\partial x_4} - \frac{\partial f_4}{\partial x_5}$ can be found with negative sign at positions $_{123}$, $_{231}$ and $_{312}$ and with positive sign at positions $_{132}$, $_{213}$ and $_{321}$. The tensor contains $n(n-1)/2$ different elements apart from sign changes, each repeated $(n-2)!$ times, while the rest of the n^{n-2} elements is zero. This enormous complexity makes higher-dimensional Helmholtz analysis challenging.

3. An alternative n-dimensional Helmholtz Decomposition Theorem

In the following, we present a simpler approach to Helmholtz Decomposition of a twice continuously differentiable vector field $f(x)$ that decays faster than $1/|x|$ for $|x| \to \infty$. The basic idea is to understand the rotation in dimension n as a combination of $\binom{n}{2}$ rotations within the planes spanned by two of the Cartesian coordinates. In each of these planes of rotation, applying the $\overline{\text{curl}}$ in two dimensions is sufficient. These $\binom{n}{2}$ rotation densities form the upper triangle of a matrix. For tractability, we complete this to a $n \times n$ matrix by making it antisymmetric. It thus contains redundantly both the rotation in the x_i-x_j-plane and in the x_j-x_i-plane. Nevertheless, it has only n^2 entries, instead of n^{n-2} entries for $\overline{\text{curl}}$, and contains only the mutually different entries of this operator.

We proceed along the steps shown in Figure 1(b). Starting ❶ from $f(x)$, we calculate ❷ the scalar source density $\gamma(x)$ and n^2 basic rotation densities $\rho_{ij}(x)$ using the new differential operator

$\overline{\mathcal{D}}$, consisting of the well-known divergence div and the new operator $\overline{\mathrm{ROT}}$. Each basic rotation density $\rho_{ij}(x)$ corresponds to the rotation within the x_i-x_j-plane. The basic rotation densities form the n^2-dimensional, antisymmetric rotation density $\rho(x) = \left[\!\left[\rho_{ij}(x)\right]\!\right] = \overline{\mathrm{ROT}} f(x)$ and together with $\gamma(x) = \operatorname{div} f(x)$ the density $\phi(x) = \left\{\gamma(x), \rho(x)\right\} = \left\{\gamma(x), \left[\!\left[\rho_{ij}(x)\right]\!\right]\right\}$. The Newton Integral operator ${}^{N}\!\!\int$ convolves these densities with the fundamental solutions of the Laplace equation, yielding ❸ the scalar 'source potential' $G(x) = {}^{N}\!\!\int \gamma(x)$ and n^2 'basic rotation potentials' $R_{ij}(x) = {}^{N}\!\!\int \rho_{ij}(x)$. Similar to the densities, these n^2 scalar fields $R_{ij}(x)$ can be jointly written as antisymmetric 'rotation potential' $R(x) = \left[\!\left[R_{ij}(x)\right]\!\right]$. With a new differential operator $\mathcal{D}$, combining the well-known gradient grad with the new operator ROT, operating on the potential $F = \{G, R\}$, a rotation-free 'gradient field' $g(x) = \operatorname{grad} G(x)$ and a source-free 'rotation field' $r(x) = \mathrm{ROT}\, R(x)$ can be calculated. In sum ❹, they yield the original vector field $f(x)$.

Theorem 1 (Helmholtz Decomposition Theorem)**.** *Any twice continuously differentiable vector field $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ that decays faster than $1/|x|$ for $|x| \to \infty$ can be decomposed into two vector fields, one rotation-free and one divergence-free. With the definitions of the operators in Section 4, let*

$$G \in C^3(\mathbb{R}^n, \mathbb{R}), \qquad \textit{with } G(x) := {}^{N}\!\!\int \operatorname{div} f(x), \tag{8}$$

$$g \in C^2(\mathbb{R}^n, \mathbb{R}^n), \qquad \textit{with } g(x) := \operatorname{grad} G(x), \tag{9}$$

$$R \in C^3(\mathbb{R}^n, \mathbb{R}^{n^2}), \qquad \textit{with } R(x) := {}^{N}\!\!\int \overline{\mathrm{ROT}} f(x), \tag{10}$$

$$r \in C^2(\mathbb{R}^n, \mathbb{R}^n), \qquad \textit{with } r(x) := \mathrm{ROT}\, R(x), \tag{11}$$

then

g, the gradient of G, is rotation-free: $$0 = \overline{\mathrm{ROT}} g(x) = \overline{\mathrm{ROT}} \operatorname{grad} G(x), \tag{12}$$

r, the rotation of R, is divergence-free: $$0 = \operatorname{div} r(x) = \operatorname{div} \mathrm{ROT}\, R(x), \tag{13}$$

the potential $\{G, R\}$ is an antiderivative of f: $$f(x) = \mathcal{D}\{G(x), R(x)\}, \tag{14}$$

the Helmholtz Decomposition of f is given by: $$f(x) = g(x) + r(x). \tag{15}$$

We prove Eq. (12) as Proposition 2, Eq. (13) as Proposition 3 and Eqs. (14–15) as Proposition 9 in Section 5.

4. Definitions

The notation is that square brackets $[f_k]$ indicate a vector, double square brackets $[\![R_{ij}]\!]$ a $n \times n$ matrix and curly brackets $\left\{\gamma, [\![\rho_{ij}]\!]\right\}$ an object of dimension $1 + n^2$. For comparison, the table in the Appendix A summarizes the definitions and important properties of the operators and variables, and shows their similarities to the well-known decomposition in dimension 3.

4.1. Densities, density rotation operator $\overline{\mathrm{ROT}}$, and density operator $\overline{\mathcal{D}}$

Let $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ be a twice continuously differentiable vector field that decays faster than $1/|x|$ for $|x| \to \infty$.

$$f(x) = \left[f_k(x); 1 \le k \le n\right] = \left[f_1(x), \ldots, f_n(x)\right]. \tag{16}$$

To get from the vector field f to the source and rotation densities, we start with the Jacobian Matrix J of f. J can be decomposed into a symmetric part S and an antisymmetric part A:

$$J = \left[\!\left[J_{ij}\right]\!\right] = \left[\!\left[\frac{\partial f_i}{\partial x_j}\right]\!\right] = S + A \quad \text{with} \quad A = \frac{J + J^\top}{2} \quad \text{and} \quad S = \frac{J - J^\top}{2}. \tag{17}$$

We define the **'scalar source density'** $\gamma \in C^1(\mathbb{R}^n, \mathbb{R})$ analogously to $\mathbb{R}^3$ by using only the diagonal of the symmetric part:

$$\gamma(x) := \operatorname{Tr} J = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = \operatorname{div} f(x). \tag{18}$$

We define the **'basic rotation density operator'** $\overline{\mathrm{ROT}_{ij}} \colon C^1(\mathbb{R}^n, \mathbb{R}^n) \mapsto C^0(\mathbb{R}^n, \mathbb{R})$ for $1 \leq i, j \leq n$ as -2 times the i-j-element of the antisymmetric part A:

$$\overline{\mathrm{ROT}_{ij}} f(x) = -2A_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}. \tag{19}$$

This generalizes the two-dimensional $\overline{\mathrm{curl}}$ in the x_1-x_2-plane of Eq. (5) given by $\overline{\mathrm{curl}} f = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$ to the rotation in the x_i-x_j-plane. To avoid confusion with $\overline{\mathrm{curl}}$, we use $\overline{\mathrm{ROT}}$ for rotation of vector fields – and later define ROT_{ij} analogously to curl operating on scalar fields.

We define the **'rotation density operator'** $\overline{\mathrm{ROT}} \colon C^1(\mathbb{R}^n, \mathbb{R}^n) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^{n^2})$ as an antisymmetric operator containing all the basic rotation density operators:

$$\overline{\mathrm{ROT}} f(x) := \left[\!\left[\overline{\mathrm{ROT}_{ij}} f(x) \right]\!\right] = \left[\!\left[\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right); 1 \leq i, j \leq n \right]\!\right]. \tag{20}$$

We define the n^2 **'basic rotational densities'** $\rho_{ij} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for $1 \leq i, j \leq n$ and the matrix $\rho \in C^1(\mathbb{R}^n, \mathbb{R}^{n^2})$ containing these n^2 densities as the rotation density operators applied to the vector field $f(x)$:

$$\rho_{ij}(x) := \overline{\mathrm{ROT}_{ij}} f(x), \tag{21}$$

$$\rho(x) := \left[\!\left[\rho_{ij}(x) \right]\!\right] = \left[\!\left[\rho_{ij}(x); 1 \leq i, j \leq n \right]\!\right] = \left[\!\left[\overline{\mathrm{ROT}_{ij}} f(x); 1 \leq i, j \leq n \right]\!\right] = \overline{\mathrm{ROT}} f(x). \tag{22}$$

As $\overline{\mathrm{ROT}_{ij}}$ and ρ_{ij} are antisymmetric, they in fact only contain $\binom{n}{2}$ independent elements, one for each of the $\binom{n}{2}$ coordinate planes.

We define the **'density derivative'** $\overline{\mathcal{D}} \colon C^1(\mathbb{R}^n, \mathbb{R}^n) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^{1+n^2})$ of any vector field f that combines div and ROT into one operator, and define the **'density'** $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^{1+n^2})$ that combines the source and rotation densities:

$$\phi(x) := \left\{ \gamma(x), \rho(x) \right\} = \left\{ \gamma(x), \left[\!\left[\rho_{ij}(x) \right]\!\right] \right\} = \overline{\mathcal{D}} f(x) := \left\{ \operatorname{div} f(x), \overline{\mathrm{ROT}} f(x) \right\}. \tag{23}$$

4.2. Source and Rotation Potentials, and Newton Integration

The next step derives the potentials starting from the densities. We define the one-dimensional **'source potential'** $G \in C^3(\mathbb{R}^n, \mathbb{R})$ and the n^2 **'basic rotation potentials'** $R_{ij} \in C^3(\mathbb{R}^n, \mathbb{R})$ for $1 \leq i, j \leq n$, corresponding to the coordinate plane spanned by x_i and x_j. Similar to the three-dimensional case, the potentials can be derived using the **'Newton potential operator'** $\overset{N}{\int} \colon C^0(\mathbb{R}^n, \mathbb{R}) \mapsto C^2(\mathbb{R}^n, \mathbb{R})$, the convolution of the respective densities with the fundamental solutions of the Laplace equation $\Delta K(x) = 0$.

$$G(x) := \overset{N}{\int} \gamma(x) := \int_{\mathbb{R}^n} K(|x - \xi|)\, \gamma(\xi)\, d\xi^n, \tag{24}$$

$$R_{ij}(x) := \overset{N}{\int} \rho_{ij}(x) = \int_{\mathbb{R}^n} K(|x - \xi|)\, \rho_{ij}(\xi)\, d\xi^n, \tag{25}$$

with the Green kernel $K(x)$, using $V_n = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ the volume of a unit n-ball and $\Gamma(x)$ the gamma function,

$$K(|x-\xi|) = \begin{cases} \frac{1}{2\pi}\log|x-\xi| & n=2, \\ \frac{1}{n(2-n)V_n}|x-\xi|^{2-n} & n \neq 2. \end{cases}$$

We call this convolution integral **'Newton Integral'** and the resulting potentials **'Newton potentials'**. Note that compared to the case in $\mathbb{R}^3$, we use a different sign convention: $G(x) = -\Phi(x)$. The condition that $f(x)$ decays faster than $1/|x|$ for $|x| \to \infty$ implies that the densities decay faster than $1/|x|^2$ for $|x| \to \infty$, which guarantees the existence of the Newton Integrals.

We define the **'rotation potential'** $R \in C^3(\mathbb{R}^n, \mathbb{R}^{n^2})$ as matrix containing the n^2 basic rotation potentials R_{ij}:

$$R(x) := \left[\!\left[R_{ij}(x) \right]\!\right] = \left[\!\left[R_{ij}(x); 1 \le i,j \le n \right]\!\right] = \left[\!\left[{}^{N}\!\!\int \rho_{ij}(x); 1 \le i,j \le n \right]\!\right] = {}^{N}\!\!\int \rho(x). \tag{26}$$

Here, the Newton potential operator ${}^{N}\!\!\int$ is applied in each component. The rotation potential $R(x)$ is antisymmetric, thus $R_{ij}(x) = -R_{ji}(x)$ and $R_{ii}(x) = 0$, and it therefore contains $\binom{n}{2}$ distinct potentials.

The **'potential'** $F \in C^3(\mathbb{R}^n, \mathbb{R}^{1+n^2})$ combines all these potentials into one object of dimension $1+n^2$:

$$F(x) := \left\{ G(x), R(x) \right\} = {}^{N}\!\!\int \phi(x) = {}^{N}\!\!\int \overline{\mathcal{D}} f(x). \tag{27}$$

We know from the theory of the Poisson equation (Gilbarg and Trudinger, 1977) that $\Delta {}^{N}\!\!\int Q(x) = Q(x)$ for $Q \in C(\mathbb{R}^n, \mathbb{R}^m)$, which implies:

$$\Delta G(x) = \gamma(x), \qquad \Delta R_{ij}(x) = \rho_{ij}(x)\ \forall\, i,j, \qquad \Delta R(x) = \rho(x), \qquad \Delta F(x) = \phi(x). \tag{28}$$

4.3. Rotation operator ROT, derivative of the potential $\mathcal{D}$

We define the **'basic rotation operator'** $\mathrm{ROT}_{ij}\colon C^1(\mathbb{R}^n, \mathbb{R}) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$ operating on the basic rotation potential R_{ij} with $1 \le i,j \le n$ as

$$\begin{aligned} \mathrm{ROT}_{ij}\, R_{ij}(x) :=& \left[0,\ldots,0,-\frac{\partial R_{ij}}{\partial x_j},0,\ldots,0,+\frac{\partial R_{ij}}{\partial x_i},0,\ldots,0 \right] \\ =& \left[\delta_{jk}\frac{\partial R_{ij}}{\partial x_i} - \delta_{ik}\frac{\partial R_{ij}}{\partial x_j}; 1 \le k \le n \right], \end{aligned} \tag{29}$$

with the Kronecker delta $\delta_{ik} = 1$ if $i = k$ and 0 otherwise. This operator is a generalization of curl operating on a scalar field in the two-dimensional case of rotations within the x_1-x_2-plane in Eq. (4) given by $\left[\delta_{2k}\frac{\partial R}{\partial x_1} - \delta_{1k}\frac{\partial R}{\partial x_2}; 1 \le k \le 2\right]$. It operates in the x_i-x_j-plane, therefore the non-zero terms in this n-dimensional vector are located at positions i and j, instead of 1 and 2 in the 2-dimensional case. Note that $\mathrm{ROT}_{ij}\, R_{ij}(x) = \mathrm{ROT}_{ji}\, R_{ji}(x)$, yielding a symmetric object.

We define the **'rotation operator'** $\mathrm{ROT}\colon C^1(\mathbb{R}^n, \mathbb{R}^{n^2}) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$ for each rotation potential $R(x) = \left[\!\left[R_{ij}(x) \right]\!\right]$ as superposition of the $\binom{n}{2}$ basic rotations in the coordinate planes. As the antisymmetric potential $R(x)$ of dimension n^2 contains each of the $\binom{n}{2}$ basic rotations twice, we have to devide the result by 2.

$$\mathrm{ROT}\, R(x) := \frac{1}{2}\sum_{i,j=1}^{n} \mathrm{ROT}_{ij}\, R_{ij}(x) = \left[-\sum_{m=1}^{n} \frac{\partial R_{km}}{\partial x_m}; 1 \le k \le n \right]. \tag{30}$$

Note that this is identical to $-\operatorname{div} R_{km}$ for an antisymmetric second-rank tensor.

The decomposition of $f = g + r$ into a rotation-free **'gradient field'** $g \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and a divergence-free **'rotation field'** $r \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ is given by:[3]

$$g(x) = \operatorname{grad} G(x), \qquad r(x) = \operatorname{ROT} R(x). \tag{31}$$

We define the **'derivative of the potential'** $\mathcal{D}\colon C^1(\mathbb{R}^n, \mathbb{R}^{1+n^2}) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$ operating on a potential $F(x) = \{G(x), R(x)\}$ as the sum of the gradient operating on the source potential $G(x)$ and the rotation operator ROT operating on the rotation potential $R(x)$:

$$\mathcal{D}F(x) = \mathcal{D}\Big\{G(x), R(x)\Big\} := \operatorname{grad} G(x) + \operatorname{ROT} R(x) = g(x) + r(x). \tag{32}$$

We call a potential $F = \{G, R\}$ **'antiderivative'** of f if $f(x) = \mathcal{D}F(x)$. Given the conditions on $f(x)$, the potential is uniquely[4] determined by Eq. (24). We will prove that Eqs. (31) and (32) satisfy the Helmholtz Decomposition Theorem.

5. Proofs of Operator Identities and Helmholtz Decomposition Theorem

We proceed by proving the conditions Eqs. (12–13) that $g(x)$ is curl-free and $r(x)$ is divergence-free. We introduce and prove Lemma 4 and three operator identities as Propositions 6–8. Equipped with Lemma 4 and Proposition 6, we prove the conditions Eqs. (14–15) that $g(x) + r(x) = \mathcal{D}F(x)$ equals the original vector field $f(x)$. This completes the proof of the Helmholtz Decomposition Theorem 1.

Proposition 2 (Eq. (12) of the Helmholtz Decomposition Theorem: $g(x) = \operatorname{grad} G(x)$ is rotation-free)**.** *For any twice continuously differentiable vector field $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ that decays faster than $1/|x|^2$ for $|x| \to \infty$, the source potential $G(x) = \mathcal{N}\!\!\int \operatorname{div} f(x)$ as defined by Eq.* (24) *and its gradient $g(x) = \operatorname{grad} G(x)$ satisfy the following identity:*

$$\overline{\operatorname{ROT}} g(x) = \overline{\operatorname{ROT}} \operatorname{grad} G(x) = 0. \tag{33}$$

Proof. Using the definition of $\overline{\operatorname{ROT}}$ in Eq. (20) and permutability of second derivatives:

$$\overline{\operatorname{ROT}} g(x) = \overline{\operatorname{ROT}} \operatorname{grad} G(x) = \overline{\operatorname{ROT}} \left[\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}\right] \tag{34}$$

$$= \left[\!\!\left[\frac{\partial}{\partial x_j}\frac{\partial G}{\partial x_i} - \frac{\partial}{\partial x_i}\frac{\partial G}{\partial x_j}; 1 \le i, j \le n \right]\!\!\right] = 0. \tag{35}$$

□

Proposition 3 (Eq. (13) of the Helmholtz Decomposition Theorem: $r(x) = \operatorname{ROT} R(x)$ is divergence-free)**.** *For any twice continuously differentiable vector field $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ that decays faster than $1/|x|^2$ for $|x| \to \infty$, the rotation potential $R(x) = \mathcal{N}\!\!\int \overline{\operatorname{ROT}} f(x)$ as defined by Eqs.* (25) *and* (26) *and its rotation $r(x) = \operatorname{ROT} R(x)$ satisfy the following identity:*

$$\operatorname{div} r(x) = \operatorname{div} \operatorname{ROT} R(x) = 0 \tag{36}$$

[3] Note that if one is not interested in the rotation potentials, $r(x)$ can simply be obtained after determining $G(x)$ and $g(x)$ by calculating $r(x) = f(x) - g(x)$, which was the approach by Stokes (1849).

[4] Note that by Liouville's theorem, if H is a harmonic function defined on all of $\mathbb{R}^n$ which is bounded above or bounded below, then H is constant, and therefore identical zero if it vanishes at infinity (Medková, 2018, p. 108). Therefore, we do not need to care about integration constants and adding harmonic functions that solve the Laplace Equation $\Delta H(x) = 0$. If the fields do not decay sufficiently fast, alternative methods to derive Newton Potentials can be found in the literature in footnote 2. They require a careful attention to boundary conditions because $G(x) = \mathcal{N}\!\!\int \Delta(G(x) + H(x))$ with any harmonic function $H(x)$.

Proof. Using definition of ROT in Eq. (30) and ROT_{ij} in Eq. (29), exchangeability of derivatives and sums, and permutability of second derivatives:

$$\operatorname{div} \mathrm{ROT}_{ij} R_{ij}(x) = \operatorname{div}\left[\delta_{jk}\frac{\partial R_{ij}}{\partial x_i} - \delta_{ik}\frac{\partial R_{ij}}{\partial x_j}; 1 \le k \le n\right] \tag{37}$$

$$= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\delta_{jk}\frac{\partial R_{ij}}{\partial x_i} - \delta_{ik}\frac{\partial R_{ij}}{\partial x_j}\right) = \sum_{k=1}^{n}\left(\frac{\partial^2 R_{ij}}{\partial x_j \partial x_i} - \frac{\partial^2 R_{ij}}{\partial x_i \partial x_j}\right) = 0. \tag{38}$$

$$\operatorname{div} r(x) = \operatorname{div} \mathrm{ROT}\, R(x) = \operatorname{div} \sum_{i,j}^{n} \tfrac{1}{2}\, \mathrm{ROT}_{ij} R_{ij}(x) \tag{39}$$

$$= \tfrac{1}{2}\sum_{i,j}^{n} \operatorname{div} \mathrm{ROT}_{ij} R_{ij}(x) = \sum_{i,j}^{n} 0 = 0. \tag{40}$$

□

Lemma 4 (Exchangeablity of the Newton integral with any partial derivative)**.** *For any continuously differentiable vector field $Q \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ that decays faster than $1/|x|^2$ for $|x| \to \infty$, the Newton integral as defined by Eq.* (24) *can be exchanged with any partial derivative:*

$$\frac{\partial}{\partial x_k} {}^{N}\!\!\int Q(x) = {}^{N}\!\!\int \frac{\partial Q(x)}{\partial x_k}. \tag{41}$$

In the context of this paper, Q can be G, R, R_{ij}, F, f, g, r, γ, ρ, ρ_{ij}, or ϕ.

Proof.

$$\frac{\partial}{\partial x_k} {}^{N}\!\!\int Q(x) = \frac{\partial}{\partial x_k} \int_{\mathbb{R}^n} K(|x-\xi|) Q(\xi)\, d\xi^n = \int_{\mathbb{R}^n} Q(\xi)\left(\frac{\partial}{\partial x_k} K(|x-\xi|)\right) d\xi^n \tag{42}$$

$$= \int_{\mathbb{R}^n} Q(\xi)\left(-\frac{\partial}{\partial \xi_k} K(|x-\xi|)\right) d\xi^n \tag{43}$$

using integration by parts in the k-component

$$= -\int_{\mathbb{R}^{n-1}}\left(Q(\xi)K(|x-\xi|)\Big|_{\xi_k=-\infty}^{\infty} - \int_{\xi_k=-\infty}^{\infty} \frac{\partial Q(\xi)}{\partial \xi_k} K(|x-\xi|)\, d\xi_k\right) d[\xi_i; i \ne k] \tag{44}$$

(for $n = 1$, the outer integral over $\mathbb{R}^0$ in the intermediate step above is omitted) and as $Q(\xi)K(|x-\xi|) \to 0$ for $\xi_k \to \pm\infty$

$$= \int_{\mathbb{R}^n} \frac{\partial Q(\xi)}{\partial \xi_k} K(|x-\xi|)\, d\xi^n = {}^{N}\!\!\int \left(\frac{\partial Q(x)}{\partial x_k}\right) \ \forall\, k. \tag{45}$$

□

Corollary 5. *Applying Lemma 4 in each component, it follows*

$$ {}^{N}\!\!\int \Delta Q(x) = \Delta {}^{N}\!\!\int Q(x) = Q(x) \qquad \text{for } Q \in C^2(\mathbb{R}^n, \mathbb{R}^m), \tag{46}$$

$$\mathcal{D} {}^{N}\!\!\int F(x) = {}^{N}\!\!\int \mathcal{D}F(x) \qquad \text{for a potential } F \in C^3(\mathbb{R}^n, \mathbb{R}^{1+n^2}), \tag{47}$$

$$\text{and} \quad \overline{\mathcal{D}} {}^{N}\!\!\int f(x) = {}^{N}\!\!\int \overline{\mathcal{D}} f(x) \qquad \text{for a vector field } f \in C^2(\mathbb{R}^n, \mathbb{R}^n). \tag{48}$$

Proposition 6 (Operator Identity: $\mathcal{D}\overline{\mathcal{D}}f = \Delta f$)**.** *For any twice continuously differentiable vector field* $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$*, the following identity holds:*

$$\mathcal{D}\overline{\mathcal{D}}f(x) = \operatorname{grad}\operatorname{div} f(x) + \operatorname{ROT}\overline{\operatorname{ROT}}f(x) = \Delta f(x). \tag{49}$$

Proof. The first part follows from the definition of $\mathcal{D}$ and $\overline{\mathcal{D}}$, and Proposition 2 and 3. By using Eq. (30) stating $\operatorname{ROT} R(x) = \left[-\sum_m \frac{\partial R_{km}}{\partial x_m}; 1 \le k \le n\right]$, it follows:

$$\begin{aligned}\mathcal{D}\overline{\mathcal{D}}f(x) &= \operatorname{grad}\operatorname{div} f + \operatorname{ROT}\left[\!\!\left[\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right); 1 \le i, j \le n\right]\!\!\right] \\ &= \left[\frac{\partial}{\partial x_k}\sum_{m=1}^{n}\frac{\partial f_m}{\partial x_m} - \sum_{m=1}^{n}\frac{\partial}{\partial x_m}\left(\frac{\partial f_m}{\partial x_k} - \frac{\partial f_k}{\partial x_m}\right); 1 \le k \le n\right] \\ &= \left[\sum_{m=1}^{n}\frac{\partial^2 f_k}{\partial x_m^2}; 1 \le k \le n\right] = [\Delta f_k; 1 \le k \le n] = \Delta f(x).\end{aligned} \tag{50}$$

The first two terms cancel out because of the symmetry of second derivatives (Schwarz's theorem) and interchange of sum and derivative. □

Proposition 7 (Operator Identity: $(-1)^n \operatorname{curl}\overline{\operatorname{curl}}f = \operatorname{ROT}\overline{\operatorname{ROT}}f$)**.** *For any twice continuously differentiable vector field* $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$*, the following identity holds:*

$$(-1)^n \operatorname{curl}\overline{\operatorname{curl}}f(x) = \operatorname{ROT}\overline{\operatorname{ROT}}f(x). \tag{51}$$

Proof. This follows from Proposition 6 and the well-known identity (De La Calle Ysern and Sabina De Lis, 2019, p. 522)

$$\Delta f(x) = \operatorname{grad}\operatorname{div} f(x) + (-1)^n \operatorname{curl}\overline{\operatorname{curl}}f(x). \tag{52}$$

□

Proposition 8 (Identity for Newton Potential F: $\overline{\mathcal{D}}\mathcal{D}F = \Delta F$)**.** *For any twice continuously differentiable vector field* $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ *that decays faster than* $1/|x|$ *for* $|x| \to \infty$*, the potential* $F(x) = {}^{N}\!\!\int \overline{\mathcal{D}}f(x)$ *as defined by Eqs.* (23) *and* (27) *satisfies the following identity:*

$$\overline{\mathcal{D}}\mathcal{D}F(x) = \Delta F(x). \tag{53}$$

Proof. Using the definition of $F(x)$, Corollary 5 stating ${}^{N}\!\!\int \overline{\mathcal{D}}f(x) = \overline{\mathcal{D}}\,{}^{N}\!\!\int f(x)$, Proposition 6 stating $\mathcal{D}\overline{\mathcal{D}} = \Delta$, the definition of $\phi(x)$ in Eq. (23), and $\Delta F(x) = \phi(x)$ from Eq. (28):

$$\overline{\mathcal{D}}\mathcal{D}F(x) = \overline{\mathcal{D}}\mathcal{D}\,{}^{N}\!\!\int \overline{\mathcal{D}}f(x) = \overline{\mathcal{D}}\mathcal{D}\overline{\mathcal{D}}\,{}^{N}\!\!\int f(x) = \overline{\mathcal{D}}\Delta\,{}^{N}\!\!\int f(x) = \overline{\mathcal{D}}f(x) = \phi(x) = \Delta F(x). \tag{54}$$

□

Note: While Proposition 6 is an operator identity for any field $f(x)$, this derivation is valid only if $F(x)$ was constructed following Eqs. (23) and (27). For a general function $Q \in C^3(\mathbb{R}^n, \mathbb{R}^{1+n^2})$, $\overline{\mathcal{D}}\mathcal{D}Q(x)$ may differ from $\Delta Q(x)$.

Proposition 9 (Eqs. (14–15) of the Helmholtz Decomposition Theorem: $g(x) + r(x) = f(x)$)**.** *For any twice continuously differentiable vector field* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *that decays faster than* $1/|x|$ *for* $|x| \to \infty$*, the source potential* $G(x) = {}^{N}\!\!\int \operatorname{div} f(x)$ *as defined by Eq.* (24) *and its gradient* $g(x) = \operatorname{grad} G(x)$*, the rotation potential* $R(x) = {}^{N}\!\!\int \overline{\mathrm{ROT}} f(x)$ *as defined by Eqs.* (25) *and* (26) *and its rotation* $r(x) = \mathrm{ROT}\, R(x)$*, and the potential* $F(x) = \{G(x), R(x)\}$ *as defined by Eq.* (27) *satisfy the following identity:*

$$g(x) + r(x) = \operatorname{grad} G(x) + \mathrm{ROT}\, R(x) = \mathcal{D}F(x) = f(x). \tag{55}$$

Proof. The first equalities follow immediately from the definitions. The last can be derived starting with the definition of $F(x)$, applying Corollary 5 and Proposition 6:

$$\mathcal{D}F(x) = \mathcal{D}\, {}^{N}\!\!\int \overline{\mathcal{D}} f(x) = \mathcal{D}\overline{\mathcal{D}}\, {}^{N}\!\!\int f(x) = \Delta\, {}^{N}\!\!\int f(x) = f(x). \tag{56}$$

□

Propositions 2, 3, and 9 prove Eqs. (12–15) and yield the proof of the Helmholtz Decomposition Theorem.

6. Conclusions

In this paper, we have introduced differential operators $\overline{\mathrm{ROT}}$, ROT and the generalized derivatives $\overline{\mathcal{D}}$ and $\mathcal{D}$, such that for any twice continuously differentiable vector field $f(x)$ in $\mathbb{R}^n$ that decays faster then $1/|x|$ for $x \to \infty$, a scalar source potential $G(x)$ and an antisymmetric rotation potential $R(x) = R_{ij}(x)$ with $\binom{n}{2}$ distinct entries can be calculated as convolutions of the density $\phi(x) = \overline{\mathcal{D}} f(x)$ with the fundamental solutions of the Laplace equation. The joint potential $F(x) = \{G(x), R(x)\}$ is an antiderivative of $f(x)$, such that applying the differential operator $\mathcal{D}$ to this potential provides a decomposition of $f(x)$ into a rotation-free 'gradient field' $g(x) = \operatorname{grad} G(x)$ and a source-free 'rotation field' $r(x) = \mathrm{ROT}\, R(x)$:

$$f(x) = \mathcal{D}F(x) = \mathcal{D}\, {}^{N}\!\!\int \overline{\mathcal{D}} f(x) = \mathcal{D}\, {}^{N}\!\!\int \phi(x) = g(x) + r(x). \tag{57}$$

This generalizes the Helmholtz Decomposition to $\mathbb{R}^n$ without the need for differential forms and the complicated operator curl and facilitates its application to high-dimensional dynamic systems. The potentials correspond to 'antiderivatives' of the gradient and rotation differential operators, providing a generalization of the fundamental theorem of calculus in $\mathbb{R}^n$ that links differential and integral calculus.

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A. Comparison of the Helmholtz Decomposition, operators and variables in dimension n with dimension 3

Variable or operator	dimension n	dimension 3	
source potential $G \in C^3(\mathbb{R}^n, \mathbb{R})$	$G(x)$	$\Phi(x) = -G(x)$	(58)
basic rotation potentials $R_{ij}, R_i \in C^3(\mathbb{R}^n, \mathbb{R})$	$R_{ij}(x) = -R_{ji}(x); 1 \le i, j \le n,$ $\binom{n}{2}$ distinct entries	$R_1, R_2, R_3,$ $\binom{n}{2} = 3$ distinct entries	(59)
rotation potential $R \in C^3(\mathbb{R}^n, \mathbb{R}^{n^2})$	$R(x) = [\![R_{ij}(x); 1 \le i, j \le n]\!]$ $n \times n$ matrix	$R(x) = A(x) = [R_1, R_2, R_3]$	(60)
potential $F \in C^3(\mathbb{R}^n, \mathbb{R}^{1+n^2})$	$F(x) = \{G(x), R(x)\}$	$F(x) = \{G(x), R(x)\}$	(61)
basic rotation operator $\mathrm{ROT}_{ij} \colon C^1(\mathbb{R}^n, \mathbb{R}) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$	$\mathrm{ROT}_{ij}\, R_{ij} := \left[\delta_{jk} \frac{\partial R_{ij}}{\partial x_i} - \delta_{ik} \frac{\partial R_{ij}}{\partial x_j}; 1 \le k \le n\right]$		(62)
rotation operator $\mathrm{ROT}\, R \colon C^1(\mathbb{R}^n, \mathbb{R}^{n^2}) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$	$\mathrm{ROT}\, R := \sum_{1 \le i,j \le n} \frac{1}{2} \mathrm{ROT}_{ij}\, R_{ij} = \left[-\sum_m \frac{\partial R_{km}}{\partial x_m}; 1 \le k \le n\right]$	$\mathrm{curl}\, R = \left[\frac{\partial R_3}{\partial x_2} - \frac{\partial R_2}{\partial x_3}, \frac{\partial R_1}{\partial x_3} - \frac{\partial R_3}{\partial x_1}, \frac{\partial R_2}{\partial x_1} - \frac{\partial R_1}{\partial x_2}\right]$ $\mathrm{ROT}\, R = \mathrm{curl}\, R$ with $R_1 = -R_{23}; R_2 = +R_{13}; R_3 = -R_{1,2}$	(63)
force f, gradient force g and rotation force r $f, g, r \in C^2(\mathbb{R}^n, \mathbb{R}^n)$	$f(x) = g(x) + r(x)$ $g(x) = \mathrm{grad}\, G(x); \quad r(x) = \mathrm{ROT}\, R(x)$	$f(x) = g(x) + r(x)$ $g(x) = \mathrm{grad}\, G(x); \quad r(x) = \mathrm{curl}\, R(x)$	(64) (65)
gradient fields are rotation-free	$\overline{\mathrm{ROT}} g(x) = \overline{\mathrm{ROT}}\, \mathrm{grad}\, G(x) = 0$	$\overline{\mathrm{curl}} g(x) = \overline{\mathrm{curl}}\, \mathrm{grad}\, G(x) = 0$	(66)
rotational fields are divergence-free	$\mathrm{div}\, r(x) = \mathrm{div}\, \mathrm{ROT}\, R(x) = 0$	$\mathrm{div}\, r(x) = \mathrm{div}\, \mathrm{curl}\, R = 0$	(67)
derivative of a potential $\mathcal{D} \colon C^1(\mathbb{R}^n, \mathbb{R}^{1+n^2}) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^n)$	$f(x) = \mathcal{D}F(x) := \mathrm{grad}\, G(x) + \mathrm{ROT}\, R(x)$	$f(x) = \mathcal{D}F(x) := -\,\mathrm{grad}\, \Phi(x) + \mathrm{curl}\, R(x)$	(68)
scalar source density γ $\gamma \in C^1(\mathbb{R}^n, \mathbb{R})$	$\gamma(x) = \mathrm{div}\, f(x) = \Delta G(x)$	$\gamma(x) = \mathrm{div}\, f(x) = \Delta G(x)$	(69)
basic rotation density op. $\overline{\mathrm{ROT}}_{ij} \colon C^1(\mathbb{R}^n, \mathbb{R}^n) \mapsto C^0(\mathbb{R}^n, \mathbb{R})$	$\overline{\mathrm{ROT}}_{ij} f := \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right)$		(70)
rotation density operator $\overline{\mathrm{ROT}} \colon C^1(\mathbb{R}^n, \mathbb{R}^n) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^{n^2})$	$\overline{\mathrm{ROT}} f(x) := [\![\overline{\mathrm{ROT}}_{ij} f(x)]\!] = \left[\!\!\left[\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right); 1 \le i, j \le n\right]\!\!\right]$	$\overline{\mathrm{curl}} f = \left[\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right]$	(71)
basic rotation density $\rho_{ij}, \rho_i \in C^1(\mathbb{R}^n, \mathbb{R})$	$\rho_{ij}(x) = -\rho_{ji}(x); 1 \le i, j \le n,$ $\binom{n}{2}$ distinct entries	$\rho_1(x), \rho_2(x), \rho_3(x)$ $\binom{n}{2} = 3$ distinct entries	(72)
rotation density ρ $\rho \in C^1(\mathbb{R}^n, \mathbb{R}^{n^2})$	$\rho(x) = [\![\rho_{ij}(x); 1 \le i, j \le n]\!] = \overline{\mathrm{ROT}} f(x) = \overline{\mathrm{ROT}}\, \mathrm{ROT}\, R$ $n \times n$ matrix	$\rho(x) = [\rho_1, \rho_2, \rho_3] = \overline{\mathrm{curl}} f(x) = \overline{\mathrm{curl}}\, \mathrm{curl}\, R(x)$ identify: $\rho_1 = \rho^{2,3}; \rho_2 = -\rho^{1,3}; \rho_3 = \rho^{1,2}$	(73)
density derivative $\overline{\mathcal{D}} \colon C^1(\mathbb{R}^n, \mathbb{R}^n) \mapsto C^0(\mathbb{R}^n, \mathbb{R}^{1+n^2})$	$\overline{\mathcal{D}} f(x) := \left(\mathrm{div}\, f(x), \overline{\mathrm{ROT}} f(x)\right)$	$\overline{\mathcal{D}} f(x) := \left(\mathrm{div}\, f(x), \overline{\mathrm{curl}} f(x)\right)$	(74)
density $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^{1+n^2})$	$\phi(x) = (\gamma(x), \rho(x)) = \left(\gamma(x), [\![\rho_{ij}(x)]\!]\right) = \overline{\mathcal{D}} f(x)$	$\phi(x) = (\gamma(x), \rho(x)) = (\gamma(x), [\rho_1(x), \rho_2(x), \rho_3(x)]) = \overline{\mathcal{D}} f(x)$	(75)
Newton potential operator ${}^{N}\!\!\int \colon C^0(\mathbb{R}^n, \mathbb{R}) \mapsto C^2(\mathbb{R}^n, \mathbb{R})$	$G(x) = {}^{N}\!\!\int \gamma(x) = \int_{\mathbb{R}^n} K(\lvert x - \xi \rvert)\, \gamma(\xi)\, d\xi^n$ $R_{ij}(x) = {}^{N}\!\!\int \rho_{ij}(x) = \int_{\mathbb{R}^n} K(\lvert x - \xi \rvert)\, \rho_{ij}(\xi)\, d\xi^n$	$\Phi(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\gamma(\xi)}{\lvert x - \xi \rvert}\, d\xi^3$ $R(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\rho(\xi)}{\lvert x - \xi \rvert}\, d\xi^3$	(76) (77)

with the Green kernel $K(|x - \xi|) = \begin{cases} \frac{1}{2\pi} \log |x - \xi| & n = 2 \\ \frac{1}{n(2-n)V_n} |x - \xi|^{2-n} & n \neq 2, \end{cases}$ and $V_n = \pi^{\frac{n}{2}} / \Gamma(\frac{n}{2})$ the volume of a unit n-ball, $V_3 = 4\pi/3$